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Conversion between triangular and rectangular Bézier patches

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In memory of P. Bézier

Abstract

This paper presents an explicit formula that converts a triangular Bézier patch of degree n to a degenerate rectangular Bézier patch of degree $n \times n$ by reparametrization. Based on this formula, we develop a method for approximating a degenerate rectangular Bézier patch by three nondegenerate Bézier patches; more patches can be introduced by subdivision to meet a user-specified error tolerance. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Bézier surfaces; Degree elevation; Subdivision; Conversion

Bézier surfaces are among the most widely used techniques in CAGD (Farin, 1990; Warren, 1992; Hoschek and Lasser, 1993), and they serve as a fundamental modeling tool in many CAD system. The two main types of Bézier surface are rectangular and triangular Bézier patches, which are respectively defined in terms of the univariate Bernstein polynomial $B_i^n(s) = \binom{n}{i} s^i (1-s)^{n-i}$ and the bivariate Bernstein polynomial $B^n_{i,j,k}(u,v,w) = \binom{n}{i,j,k} u^i v^j w^k$, where $u+v+w=1$. A triangular Bézier patch of degree n with control points $T_{i,j,k}$ is defined by

$$T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k} B^n_{i,j,k}(u,v,w), \quad u,v,w \geqslant 0,\ u+v+w=1, \tag{1}$$

and a rectangular Bézier patch of degree $n \times m$ with control points P_{ij} is represented by

$$P(s,t) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} B_i^n(s) B_j^m(t), \quad 0 \leqslant s,t \leqslant 1. \tag{2}$$

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Due to their different geometric properties and incompatibility, it's difficult to use both kinds of patches in the same CAD system. Hence the conversion of one type to the other has aroused the interests of some researchers (Goldman and Filip, 1987; Brueckner, 1980; Hu, 1996). The following theorem shows that a triangular Bézier patch can be converted to an equivalent *degenerate* rectangular Bézier patch, whose control points are computed by degree elevating some Bézier curves. Here, a rectangular patch in which one of its edges collapsed into a point is called a degenerate patch.

Theorem 1. *A degree n triangular Bézier patch $T(u,v,w)$ can be represented as a degenerate rectangular Bézier patch of degree $n \times n$:*

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{ij} B_i^n(s) B_j^n(t), \quad 0 \leqslant s,t \leqslant 1, \tag{3}$$

where the control points P_{ij} are determined by

$$\begin{pmatrix} P_{i0} \\ P_{i1} \\ \vdots \\ P_{in} \end{pmatrix} = A_1 A_2 \cdots A_i \begin{pmatrix} T_{i0} \\ T_{i1} \\ \vdots \\ T_{i,n-i} \end{pmatrix}, \quad i = 0,1,\ldots,n, \tag{4}$$

and A_i $(i = 0,1,\ldots,n)$ are degree elevation operators in the form

$$A_k = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \ldots & 0 & 0 \\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{n-k}{n+1-k} & \frac{1}{n+1-k} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}_{(k+1)\times k}. \tag{5}$$

Proof. We apply reparametrization

$$\begin{cases} s = u, \\ t = \frac{v}{1-u} = \frac{v}{v+w} \end{cases}$$

to $T(u,v,w)$. Thereby, the domain of the triangular patch $\{(u,v) \mid 0 \leqslant u, v, u+v \leqslant 1\}$ is transformed into square $[0,1] \times [0,1]$, and

$$\begin{aligned} B_{i,j,k}^n(u,v,w) &= \binom{n}{i,j,k} u^i v^j w^k = \binom{n}{i,j,k} u^i v^j w^k \frac{(1-u)^{n-i}}{(v+w)^{j+k}} \\ &= \binom{n}{i}\binom{n-i}{j} u^i (1-u)^{n-i} \left(\frac{v}{v+w}\right)^j \left(1 - \frac{v}{v+w}\right)^k \\ &= B_i^n(s) B_j^{n-i}(t). \end{aligned} \tag{6}$$

Substituting (6) into (1), we have

$$T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k} B_{i,j,k}^n(u,v,w) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} T_{i,j,n-i-j} B_i^n(s) B_j^{n-i}(t).$$

To represent $T(u, v, w)$ as a rectangular Bézier patch of degree $n \times n$, it is only necessary to raise degree of the following Bézier curves

$$C_i(t) = \sum_{j=0}^{n-i} T_{i,j,n-i-j} B_j^{n-i}(t), \quad i = 1, 2, \ldots, n, \tag{7}$$

into degree n, and it is obvious that the control points after degree elevated $C_i(t)$ can be represented as in Eq. (4). This completes the proof.

Based on Theorem 1, it is easy to derive a recursive algorithm to compute the new control points. Fig. 1 shows the transformation of the control nets when the domain of surface is converted from triangular to rectangular.

We now consider the inverse process. From Eq. (4), to approximate a degenerate rectangular Bézier patch $P(s, t)$ of degree $n \times n$ by a triangular Bézier patch $T(u, v)$ with degree n, it suffices to reduce the degree of the following degree n Bézier curves

$$C_i(t) = \sum_{j=0}^{n} P_{ij} B_j^n(t), \quad 0 \leqslant t \leqslant 1;\ 1 \leqslant i \leqslant n, \tag{8}$$

to $n - i$. Degree reduction of Bézier curves has been extensively investigated (Hu and Jin, 1998; Eck, 1993). By using any exsisting method, we can determine the control points $T_{i,j,n-i-j}$ $(j = 0, 1, \ldots, n - i)$ of the degree-reduced curves, which are the control points of the approximate triangular patch.

In CAD applications, we often want to design a surface that has only three boundary curves. If the system only supports rectangular patches, a natural approach is to represent the surface as a degenerate rectangular patch with a collapsed edge. However, since degenerate patches are undesirable in CAGD (Farin, 1989), we face the problem of converting this degenerate rectangular Bézier patch to nondegenerate ones.

We firstly approximate the degenerate rectangular patch by a triangular patch using above method, and converting the triangular patch to three nondegenerate rectangular Bézier patches using an explicit formula we presented in (Hu, 1996). Fig. 2 shows the results of the proposed method: (a) is the original degenerate rectangular patch, (b) is the approximated triangular patch, and (c) is the three nondegenerate rectangular patches.

Furthermore, the proposed method can be combined with subdivision to produce approximate patches that are within some preset error tolerance. Similar to the analysis

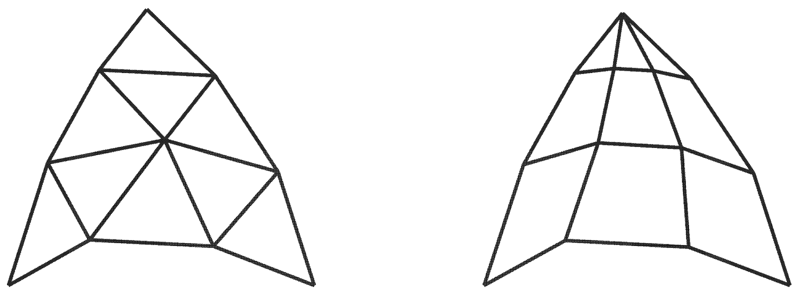


Fig. 1. Transformation of the control net under reparametrization.

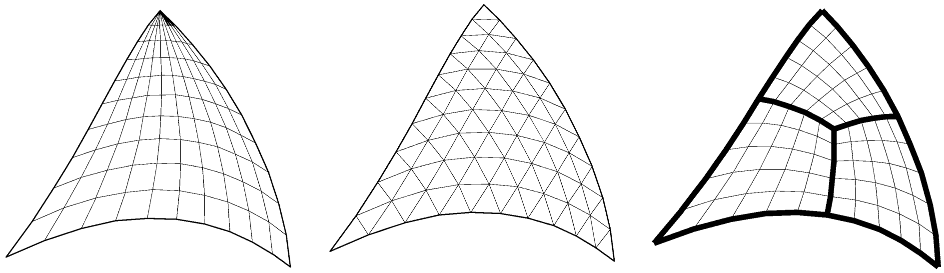
of the algorithm in (Hu and Jin, 1998), we can prove the error of the degree reduction is dependent on

$$\left\| \sum_{j=0}^{k} (-1)^j \binom{k}{j} P_{ij} \right\| \quad i = 1, 2, \ldots, n;\ k = n - i + 1,\ n - i + 2, \ldots, n. \tag{9}$$

A Bézier surface $P(s,t)$ with control points $P_{ij},\ 0 \leqslant i, j \leqslant n$, can be subdivided into four patches at parameter $s = \frac{1}{2}$ and $t = \frac{1}{2}$, we can prove that

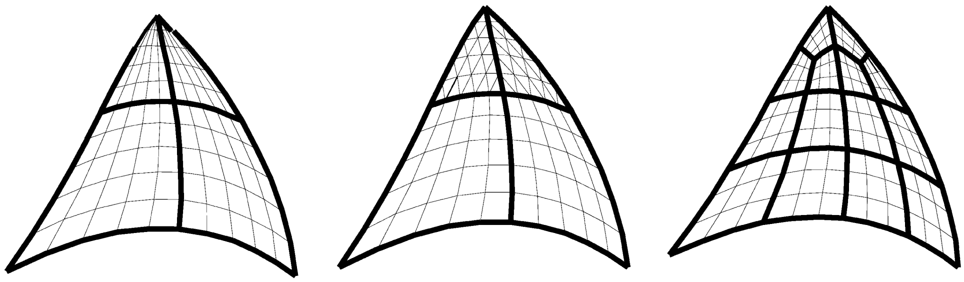
$$\left\| \sum_{j=0}^{k} (-1)^j \binom{k}{j} P_{ij}^h \right\| = \frac{1}{2^k} \left\| \sum_{j=0}^{k} (-1)^j \binom{k}{j} P_{ij} \right\|, \\ k = n - i + 1,\ n - i + 2, \ldots, n, \tag{10}$$

where $P_{ij}^h,\ 0 \leqslant i, j \leqslant n,\ h = 1, 2, 3, 4$, are control points of one of the four new surfaces after subdivision. This implies that the approximate precision of the proposed method can be improved by combining with subdivision algorithms; that is, a degenerate rectangular



(a) (b) (c)

Fig. 2. Converting a degenerate rectangular patch to three nondegenerate rectangular patches.



(a) (b) (c)

Fig. 3. Combine with subdivision to approximate triangular patch with nondegenerate rectangular patches.

Bézier surface can be first subdivided into more patches, to within a error tolerance set by the user, before the approximate triangular are determined.

Fig. 3 show effect of the approximation method combined with subdivision. Fig. 3(a) shows the original patch, which is subdivided into four patches, two of which are degenerate. Fig. 3(b) shows the degenerate patches approximated by triangular Bézier patches, and Fig. 3(c) shows 14 nondegenerate rectangular Bézier patches that approximate the original degenerate patch.

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References

Brueckner, I., 1980. Construction of Bézier points of quadrilaterals from Bézier triangles. Computer-Aided Design 12, 21–24.

Eck, M., 1993. Degree reduction of Bezier curves. Computer Aided Geometric Design 10, 237–251.

Farin, G., 1990. Curve and Surface for CAGD: A Practical Guide. Academic Press, New York.

Farin, G., 1989. Trends in curve and surface design. Computer Aided Design 21, 293–296.

Goldman, R., Filip, D., 1987. Conversion from Bézier rectangles to Bézier triangles. Computer-Aided Design 19, 25–27.

Hoschek, L., Lasser, D., 1993. Fundamentals of Computer Aided Geometric Design. A K Peters.

Hu, S.-M., 1996. Conversion of a triangular Bézier patch into three rectangular Bézier patches. Computer Aided Geometric Design 13, 219–226.

Hu, S.-M., Jin, T.-G., et al., 1998. Approximate degree reduction of Bézier curves. Tsinghua Science and Technology 3, 997–1000.

Warren, J., 1992. Creating multisided rational Bézier surfaces using base points. ACM Transactions on Graphics 11, 127–139.